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A Perturbation Property of W* Algebras.

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ABSTRACT

Let M be a Banach algebra with identity.

<u>Theorem A</u>. If $A \in M$ is contained in a proper two sided ideal of M, then A does not perturb any $T \in M$ (A perturbs T if the spectrum of $T+A$ is disjoint from the spectrum of T). The converse of Theorem A was proved in 1970 by Dyer, Porcelli and Rosenfeld [2] in the case when M is $\underline{B}(H)$, the algebra of all bounded operator on a Hilbert space H. They also conjectured the converse for a II_1 factor. In this paper we prove that the converse is true for any W* algebra (Theorem C) and that it is false for the shift algebra, the subalgebra of $\underline{B}(H)$ generated by the unilateral shift (Theorem 3).

INTRODUCTION

In Chapter I the general Dyer, Porcelli, Rosenfeld problem is discussed for Banach algebras with identity. Theorem A is proved in this setting and its converse is proved for commutative algebras (Theorem 2.2) and for the Calkin algebra (Theorem 2.3). The idea for the commutative case (namely that if A is invertible then it suffices to take $T = 0$) is implicit in the results for the W* algebras and for the shift algebra.

Chapters II - IV are concerned with the proof of Theorem C, that every $A \in M$ which is contained in no proper two sided ideal of M is perturbing where M is a W* algebra, a C* algebra with a predual (Sakai [6]). Theorem C follows from Theorem B, a stronger result for a Hermitian (self-adjoint) element A of the W* algebra, M. The proof of Theorem B is in three steps. First A is approximated by a Hermitian element B in a type I subalgebra (Chapter IV). Then a Hermitian element T such that iT perturbed by B is constructed in the type I subalgebra, and finally the upper semi-continuity of the spectrum implies that iT is perturbed by A (Chapter V).

Chapter II outlines the proof of Theorem B for the finite factors. Lemma 2.1 introduces the matrices T_n which will be added together to produce T. For $n = 2$, T_n is the matrix used by Dyer, Porcelli, Rosenfeld and Halmos in their proof of Theorem B for $\underline{B}(H)$. Using T_n, we immediately have a new proof of Theorem B for a I_n factor $(n < \infty)$. The proof of Theorem B in the case of a II_1 factor motivates the constructions of Chapters III, IV, and V. Combining the proofs of Theorems C and A with those in Chapter II, a new proof of the simplicity of finite factors may be obtained. (A C* algebra is simple if it has no proper closed two sided ideals.)

In Chapter III we introduce a new tool which generalizes a concept from the classical direct integral decomposition theory for W* algebras on separable Hilbert spaces to arbitrary W* algebras. We discuss its properties and we use it to generalize the crucial step in the proof of Theorem B for a II_1 factor to a finite W* algebra.

In Chapter IV we prove an approximation property for Hermitian elements of W* algebras. Namely, every Hermitian element contained in no proper two sided ideal of a

W* algebra M can be approximated by a Hermitian element in a subalgebra which is a finite direct sum of type I_n algebras in a special way (Lemma 4.4). Building up to this, we obtain a characterization of Hermitian elements in no proper two sided ideal (Lemma 4.1) and two characterizations of projections in no two sided ideal (Lemmas 4.1 and 4.2). The proof of Theorem B goes through for any C* algebra satisfying an approximation property. The shift algebra, for example, has an approximation property (Theorem 2).

Chapter V begins with the proof of Theorem B for a type I_n algebra (Lemma 5.1). The idea is to take the direct integral of a direct sum of copies of the matrix T_n obtained in the proof of Theorem B for a I_n factor. Theorems B and C then follow easily. The idea in the proof of Lemma 4.1 is used to start the proof of Theorem C.

Finally Chapter VI shows that Theorem C does not generalize to the shift algebra although Theorem B does generalize to the shift algebra.

<u>Notation</u>. If x_i are elements of a W^* algebra M, $\Sigma_i x_i$ will always stand for an internal direct sum, i.e. $x_i x_j = 0$ for $i \neq j$. $\Sigma \oplus_i x_i$ will be reserved for an external direct sum, i.e., $x_i \in M_i$ and $\Sigma \oplus_i x_i \in \Sigma \oplus_i M_i$ (see page 36).

$\vee_i P_i$ and $\wedge_i P_i$ will stand for the supremum and infimum of $\{P_i\}_{i \in I}$.

$\underline{R}$, $\underline{C}$ and $\underline{R}^+$ will stand for the real numbers, the complex numbers and the extend reals $\underline{R}^+ = \underline{R} \cup \{-\infty, \infty\}$. Finally, $\underline{R}(K)$, $\underline{C}(K)$ and $\underline{R}^+(K)$ will stand for the continuous $\underline{R}$, $\underline{C}$ or $\underline{R}^+$ valued functions on a topological space K .

CHAPTER I

THE GENERAL PROBLEM

M will be a Banach algebra with identity throughout this section. An element $A \in M$ is called _perturbing_ if there exists a T in M such that the spectrum of $T + A$ is disjoint from the spectrum of T, that is if

$$\sigma(T+A) \cap \sigma(T) = \emptyset .$$

If this is the case, we say that A perturbs T.

Theorem A. If $A \in M$ is perturbing, then A is contained in no proper two sided ideal of M.

Proof. We will show the contrapositive. Suppose $A \in M$ is contained in a proper two sided ideal, J, of M. Then the closure, $\overline{J}$, of J is a proper two sided ideal of M since M has an identity. Let

$$p : M \to M/\overline{J}$$

be the natural projection. Then for all $T \in M$,

$$\sigma(T) \supset \sigma(p(T)) \neq \emptyset$$

and

$$\sigma(T+A) \supset \sigma(p(T+A)) = \sigma(p(T))$$

so that

$$\sigma(T+A) \cap \sigma(T) \neq \emptyset .$$

Hence A is not perturbing. q.e.d.

The general problem is to determine when the converse of Theorem A is ture. The converse is true if M is a W* algebra (Theorem C), but false if M is the shift algebra (Theorem 3). The next two theorems show that the converse is true for commutative algebras and for the Calkin algebra.

Theorem 2.2. If M is commutative, then every element contained in no proper two sided ideal of M is perturbing.

Proof. The elements contained in no proper two sided ideal of a commutative Banach algebra with identity are exactly the invertible ones. On the other hand, if $A \in M$ is invertible, then A perturbs the zero of M since

$$\sigma(A) \cap \{0\} = \emptyset .$$

Hence A is perturbing. q.e.d.

<u>Theorem 2.3</u>. Let N be a Banach algebra with identity such that every element contained in no proper two sided ideal of N is perturbing. Suppose M is the image of N under an algebra morphism $p:N \to M$. If every element contained in no proper two sided ideal of M can be lifted to an element contained in no proper two sided ideal of N, then every element contained in no proper two sided ideal of M is perturbing.

<u>Proof</u>. Let $A \in M$ be contained in no proper two sided ideal of M. By assumption, there exist T and B in N such that $p(B) = A$ and

$$\sigma(T+B) \cap \sigma(T) = \emptyset .$$

Now $\sigma(p(T)) \subset \sigma(T)$ and

$$\sigma(A+p(T)) \subset \sigma(B+T)$$

so that

$$\sigma(A+p(T)) \cap \sigma(p(T)) = \emptyset .$$

Hence A is perturbed by $p(T)$. q.e.d.

<u>Example</u>. Let M be the Calkin algebra and $N = \underline{B}(H)$. Then M and N fulfill the hypothesis of Theorem 2.3 and the conclusion is that the converse of Theorem A is true for the Calkin algebra.

Algebras without identity

If A is an algebra without identity, then $\sigma(A)$ is the spectrum of $A \oplus 0$ in $M \oplus \underline{C}$, the algebra with identity adjoined. Since M is an ideal in $M \oplus \underline{C}$, every element of M has zero in its spectrum. Hence an algebra without identity never has any perturbing elements.

CHAPTER II

SOLUTIONS FOR THE FINITE FACTORS

In this chapter we present the two most basic lemmas (Lemma 2.1 and 2.2) for our proof of Theorem B in Chapter IV. We use them to sketch the proof of Theorem B when M is a finite factor. In this case we prove that for every nonzero Hermitian element A of M, there exists a Hermitian $T \in M$ such that

$$\sigma(T+iA) \cap \underline{R} = \emptyset .$$

Let T_n be an $n \times n$ matrix with ones down the subdiagonal and superdiagonal and with zeros everywhere else. Let P_n be an $n \times n$ matrix with a one in the lower right hand corner and with zeros elsewhere. For $n = 3$, the pictures are

$$T_3 = \begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{matrix}$$

$$P_3 = \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{matrix} \quad .$$

($n = 1,2,\cdots$ and $T_1 = 0$ and $P_1 = 1$.)

Lemma 2.1. The spectrum of $T_n + iP_n$ has no real numbers, i.e.

$$\sigma(T_n+iP_n) \cap \underline{R} = \emptyset \ .$$

Proof. Let $p_n(t)$ be the determinant of $T_n + iP_n - t$, and let $q_n(t)$ be the determinant of $T_n - t$. Since the determinant is linear in the last row,

$$p_n(t) = q_n(t) + iq_{n-1}(t)$$

where we set $q_0(t) = 1$.

Expand $q_n(t)$ along the first row and column to obtain the recursion relation

$$q_n(t) = -tq_{n-1}(t) - q_{n-2}(t)$$

with initialization $q_0(t) = 1$ and $q_1(t) = -t$.

Now suppose there exists $t \in \sigma(T_n + iP_n) \cap \underline{R}$. Then $p_n(t) = 0$ and hence $q_n(t) = q_{n-1}(t) = 0$. Hence $q_0(t) = 0$ by the recursion relation, but this contradicts $q_0(t) = 1$. Hence $\sigma(T_n + iP_n) \cap \underline{R} = \emptyset$. q.e.d.

Remark. An explicit formula for the $q_n(t)$ of the preceeding proof may be found by the use of a ''generating'' function, i.e. by finding an analytic function $f(z,t)$ such that

$$\frac{d^n f}{dz^n}(0,t) = q_n(t) \ .$$

For $|t| < 2$, this yields

$$q_n(t) = \frac{\sin\left((n+1)\cos^{-1}\left(\frac{-t}{2}\right)\right)}{\sin\left(\cos^{-1}\left(\frac{-t}{2}\right)\right)}$$

where $0 < \cos^{-1}(x) < \pi$.

Proof of Theorem B for a I_n factor.

Let $0 \neq A = A^* \in M = \underline{B}(\underline{C}^n)$. Pick a basis so that A is represented by a diagonal matrix. Let $0 \neq a \in \sigma(A)$. Then A is the direct sum of aP_n and A' ,

$$A = aP_n \oplus A'$$

where A' is invertible, $n < k$, and $n - 1$ is the dimension of $\ker A$. Since A' is invertible, $0 \notin \sigma(A)$. Let $T = aT_n \oplus 0$. Then

$$\sigma(T+iA) = \sigma(aT_n + iaP_n) \cup \sigma(iA')$$

so that

$$\sigma(T+iA) \cap \underline{R} = \emptyset \ .$$

q.e.d.

Notation. The basic constant governing the application of Lemma 2.1 is

$$d_n = \sup_t \|(T_n+iP_n-t)^{-1}\|$$

where $t \in R$. $d_n < \infty$ by Lemma 2.1.

The reason d_n is so important is the following lemma (Halmos [2], pages 53, 151, 245 and 248).

Lemma 2.2. Let B and C be elements of a Banach algebra with identity. Suppose that $\sigma(B) \cap \underline{R} = \emptyset$ and that

$$\|C-B\| < \frac{1}{\sup_t \|(B-t)^{-1}\|}$$

where t varies over the real numbers, $\underline{R}$, then

$$\sigma(C) \cap \underline{R} = \emptyset .$$

Proof of Theorem B for a II_1 factor.

Let A be a non-zero Hermitian element of a II_1 factor M. Without loss of generality, we may assume that 0 and 1 spectrum of A. Let $\{P(E)\}_{E\subset\underline{R}}$ be the spectral family of A where E runs over the Borel subsets of $\underline{R}$. ''dim'' will be the von Neumann dimension function on M.

The construction of T proceeds in six steps:

1) Let m be a positive integer such that

$$\frac{\dim P[-\frac{1}{4}, \frac{1}{4}]}{m} \leq \frac{\dim P(\frac{3}{4}, 1\frac{1}{4})}{2} .$$

2) Let $\epsilon > 0$, $\epsilon < \frac{1}{4}$, and $\epsilon < \frac{3/4}{k_n}$.

3) Choose $k > \frac{1}{2\epsilon}$ and divide $P[-\epsilon,\epsilon]$ into mk equivalent and orthogonal projections

$P_{h,i}$ $(h=1,\cdots,m, i=1,\cdots,k)$

$$P[-\epsilon,\epsilon] = \Sigma_{h,i} P_{h,i} .$$

4) Divide $P[\frac{3}{4}, 1\frac{1}{4}]$ into 2k equivalent and orthogonal projections $P_i'(i=1,\cdots,2k)$, $P_i' \sim P_{1,1}$, so that there are 2k numbers $a_i(i=1,\cdots,2k)$ such that

i) P_i' commutes with A

ii) $\frac{3}{4} = a_0 \leq a_1 \cdots \leq a_{2k} \leq 1\frac{1}{4}$ and

iii) $P(a_{i-1},a_i) \leq P_i' \leq P[a_{i-1},a_i]$

where $i=1,\cdots,2k$.

5) Choose k of $\{P_i'\}$ such that $a_i - a_{i-1} < \epsilon$ and call them $P_{n,i}$ $(n=m+1$ and $i=1,\cdots,k)$. k such P_i' exist for otherwise

$$\Sigma_{i=1}^{2k} a_i - a_{i-1} > k\,\epsilon > \frac{1}{2}$$

which contradicts condition ii) on the a_i .

The point is that

$$\|A(\Sigma_i P_{n,i}) - \Sigma_i a_i P_{n,i}\| < \epsilon ,$$

and $A(1 - \Sigma_{h,i}P_{h,i})$ is invertible in $(1 - \Sigma_{h,i}P_{h,i})M(1 - \Sigma_{h,i}P_{h,i})$; $h=1,\cdots,n$; $i=1,\cdots,k$.

6) Let $T_{n,i}$ have the matrix of T_n in the basis $P_{1,i},\cdots,P_{n,i}$. Let

$$T = \Sigma_i T_{n,i} .$$

Now apply Lemma 2.2 with $B = T + i\Sigma a_i P_{n,i}$ and $C = T + iA$ (where $i = \sqrt{-1}$ when it is not used as an index) to obtain

$$\sigma(T+iA) \cap \underline{R} = \emptyset .$$

Lemma 2.2 applies since $\sigma(T + i\Sigma_i a_i P_{n,i}) \cap R = \emptyset$ by Lemma 2.1 and

$$\|(T-iA) - (T - i\Sigma_i \; a_i P_{n,i})\|$$
$$= \|A - \Sigma_i \; a_i P_{n,i}\| \leq \epsilon < \frac{3/4}{k_n}$$

and

$$\sup_t \|(T - i\Sigma_i \; a_i P_{n,i} - t)^{-1}\| < \frac{kn}{3/4} .$$

q.e.d.

In Chapter III we generalize step 4 by replacing the numbers a_i with functions on the maximal ideal space of M . In Chapter IV (Lemma 4.4, case III) we generalize the approximation of A by $\Sigma a_i P_{n,i}$ in step 5. Step 6 is generalized in Chapter V.

CHAPTER III

THE DIRECT INTEGRAL THEORY

In this chapter, we are going to develop a direct integral theory for the spectral family of a single Hermitian element $A = A^* \in M$ in a W* algebra M. Lemma 3 is the main result.

Fix $A = A^* \in M$ and let $\{P(F)\}_{F \subset \underline{R}}$ be its spectral family (F runs over the Borel subsets of $\underline{R}$). P is a projection valued measure, that is

1) $P(\emptyset) = 0$

2) $P(\underline{R} \setminus E) = 1 - P(E)$

3) $P(\bigcup_{n=1}^{\infty} E_n) = \bigvee_{n=1}^{\infty} P(E_n)$

4) $P(E)P(F) = P(F)P(E)$ and

5) $P(E)$ is a projection in M

for all Borel subsets E, F, E_n $(n=1,\cdots,\infty)$ of $\underline{R}$.

Let Z be the center of M and let K be the maximal ideal space of Z. Then the Gelfand transform

$$\Phi : Z \to \underline{C}(K)$$

is a W*-isomorphism from Z onto the continuous complex valued functions on K. If $z \in Z$ and $m \in K$, we will write $z(m)$ instead of $\Phi(z)(m)$.

It is well known (see Sakai, [6]) that K is a Stonean space. In other words it is a compact Hausdorff space in which the closure of every open set is clopen (both closed and open). Furthermore the projections in Z correspond to the characteristic functions χ_E of the clopen subsets E of K.

Let Ω be the inverse of Φ,

$$\Omega : \underline{C}(K) \to Z .$$

<u>Definition</u>. If $a, b \in \underline{R}^+(K)$ are two continuous functions from K into the extended real numbers $\underline{R}^+ = \underline{R} \cup \{-\infty, +\infty\}$, we define

$$P(a,b) = \vee_E \, \Omega(\chi_E) P(\max_E a \, , \, \min_E b)$$

where E runs over the clopen subsets of K. We extend this definition to ''half open intervals'' and ''closed intervals'' by

$$P[a,\infty) = 1 - P(-\infty,a)$$
$$P(-\infty,b] = 1 - P(b,\infty)$$
$$P[a,b] = P[a,\infty)P(-\infty,b]$$
$$P[a,b) = P[a,\infty)P(-\infty,b) \quad \text{and}$$

$$P(a,b] = P(a,\infty)P(-\infty,b] \ .$$

We view (a,b) , $(a,b]$, etc. as intervals in the lattice $\underline{R}^+(K)$ where

$$a \leq b \Leftrightarrow a(m) \leq b(m) \quad \text{for all} \quad m \in K \ .$$

Remark. If M is a W* algebra on a Hilbert space H , and if

$$M = \int_K^{\oplus} M_m d\mu(m)$$

is the direct integral decomposition of M over its center Z , then

$$P(a,b) = \int_K^{\oplus} P(a(m),b(m))d\mu(m) \ .$$

Hence our $P(a,b)$ is a generalization of a concept from direct integral theory on separable spaces.

We present thirteen elementary properties of $P(a,b)$ which will be used in the proof of Lemma 3.

Properties.

i) If a and b are constant functions on K , i.e., if $a,b \in \underline{R}$, then our new definitions of $P(a,b)$, $P(a,b]$, etc. agree with the old.

ii) For $a,b,c,d \in \underline{R}^+(K)$,

$$P(a,b)P(c,d) = P(c,d)P(a,b)$$

and similarly for the other invervals.

iii) $P(a,b) = P(a,\infty)P(-\infty,b)$.

<u>Proof</u>. Let $Q_E = \Omega(\chi_E)\cdot P(\max_E a,\infty)$ and $R_F = \Omega(\chi_F)\cdot P(-\infty,\min_F b)$ where E and F run over the clopen subsets of K . Then

$$P(-\infty,b)P(a,\infty) = (\vee_E Q_E) \wedge (\vee_F Q_F)$$

$$= \vee_{E,F}(Q_E \wedge Q_F)$$

$$= \vee_{E,F}\Omega(\chi_{E\cap F})P(\max_E a \, , \, \min_F b) \; .$$

Taking $E = F$, we have that

$$P(-\infty,b)P(a,\infty) \geq P(a,b) \; .$$

On the other hand,

$$\Omega(\chi_{E\cap F})P(\max_E a, \, \min_F b)$$

$$\leq \Omega(\chi_{E\cap F})P(\max_{E\cap F} a, \, \min_{E\cap F} b)$$

so that

$$P(-\infty,b)P(-\infty,a) \leq P(a,b) \; .$$

q.e.d.

Property iii) insures that all the natural algebraic properties of $P(a,b)$ will be true. For example,

$$P[a,b] = 1 - (P(-\infty,a) \vee P(b,\infty)) \; .$$

For further examples of this principle, see Properties ix, x, and xi.

iv) If $Q \in M$ is a projection which commutes with A, then we define a ''spectral family'' $\{PQ(F)\}_{F \subset \underline{R}}$ by

$$(PQ)(F) = P(F)Q .$$

Strictly speaking, PQ is not the spectral family of AQ since $PQ(R) = Q \neq 1$ but all of our constructions and arguments (especially Lemma 3) apply to PQ with 1 replaced by Q, for example

$$PQ(-\infty, b] = Q - PQ(b, \infty) .$$

In particular,

$$PQ(I) = (PQ)(I)Q$$

for all our intervals I.

Finally, we note that if $B \leq Q$ and B commutes with $(PQ)(I)$ for all intervals I, then B commutes with $P(I)$ for all intervals I and hence with A.

<u>Definition</u>. A continuous function $s \in \underline{R}^+(K)$ is called <u>simple</u> if it takes on only finitely many values, i.e., if

$$s = \Sigma_{i=1}^{n} a_i \chi_{E_i}$$

for some $a_i \in R^+ = R \cup \{-\infty,\infty\}$ and for some partition $\{E_i\}_{i=1}^n$ of K into clopen sets.

v) If $s = \sum_{i=1}^n a_i \chi_{E_i}$ is a simple function, then

$$P(s,\infty) = \sum_{i=1}^n \Omega(\chi_{E_i})P(a_i,\infty) .$$

The situation is similar for $P(I)$ when I is one of the other types of intervals.

vi) Using v) we may restate the definition of $P(a,b)$ as follows:

$$P(a,b) = \bigvee_{s,s'} P(s,s')$$

where s and s' run over the simple functions such that $a \leq s$ and $s' \leq b$. Similar results hold for the other intervals. For example

$$P[a,b) = \bigvee_{s'} P[a,s')$$

where s' is simple and $s' \leq b$. This is obtained from above by using $P[a,b) = P[a,\infty)P(-\infty,b)$.

vii) $a < b$ does not imply that $P(-\infty,a] \subset P(-\infty,b)$. Hence we must be a little careful when we take complements in vi). The result is that

$$P[a,b] = \bigwedge_{s,s'} P(s,s')$$

where s and s' vary over the simple functions such that $s < a - \epsilon$ and $s' > b + \epsilon$ for some $\epsilon > 0$.

Proof.

$$P(a,b) = \bigvee_{s,s'} P(s,s')$$

where s,s' run over the simple functions such that $a + \epsilon \leq s$ and $s' \leq b-\epsilon$ for some $\epsilon > 0$. Hence

$$P[a,b] = 1 - (P(-\infty,a) + P(b,\infty))$$
$$= 1 - (\bigvee_{s,s'} P(-\infty,s) + P(s',\infty))$$

where s and s' run over the simple functions such that $s \leq a-\epsilon$ and $s' \geq b+\epsilon$ for some $\epsilon > 0$. Hence

$$P[a,b] = \bigwedge_{s,s'} (1 - P(-\infty,s) + P(s',\infty))$$
$$= \bigwedge_{s,s'} P[s,s'] .$$

However, given $s_1 \leq a-\epsilon$ and $s_1' \geq a+\epsilon$, we can find s_2 and s_2' such that $s_1 \leq s_2 - \frac{\epsilon}{2} \leq a-\epsilon$ and $s_1' \geq s_2' + \frac{\epsilon}{2} \geq a+\epsilon$ <u>since</u> a <u>is continuous</u>. Hence

$$P[s_2,s_2'] \leq P(s_1,s_1') \leq P[s_1,s_1']$$

and

$$P[a,b] = \bigwedge_{s,s'} P(s,s')$$

as claimed. q.e.d.

<u>Definition</u>. A <u>finite</u> W* algebra is an algebra with a <u>center valued trace</u>, that is a positive linear function $^{tr}:M \to Z$ such that

a) $\|x^{tr}\| \leq \|x\|$ for all $x \in M$,

b) $(zx)^{tr} = z(x^{tr}$ for all $x \in M$, $z \in Z$,

c) $(x^*x)^{tr} = 0 \Leftrightarrow x = 0$,

d) $x \to x^{tr}$ is continuous in $\sigma(M,M_*)$, the topology induced on M by its predual, and

e) $(x^*x)^{tr} = (xx^*)^{tr}$ for all $x \in M$.

We note that for two projections P and Q in a finite W* algebra M , $P \sim Q$ if and only if $P^{tr} = Q^{tr}$ (see Sakai [6]).

viii) If M is a finite W* algebra and ${}^{tr}: M \to Z$ is the center valued trace, then

$$P(-\infty,a]^{tr} = \inf_{s} P(-\infty,s)^{tr}$$

where s runs over the simple functions which are greater than $a+\epsilon$ for some $\epsilon > 0$.

Proof. $P(-\infty,s)$ is a decreasing net with $P(-\infty,a]$ as its greatest lower bound. Hence it converges to $P(-\infty,a]$ in the $\sigma(M,M_*)$ topology (the topology induced on M by its predual). But ^{tr} is continuous in the $\sigma(M,M_*)$ topology. Hence property viii) follows from property vii).

q.e.d.

ix) $P(a,b) \leq P[a,b) \leq P[a,b]$.

Proof. $P(a,b)P[a,b)$

$$= P(a,b)(1 - P(-\infty,a)P(-\infty,b)$$

$$= P(a,b) - P(a,b)P(b,a) = P(a,b)$$

and

$$P[a,b)P[a,b]$$

$$= P[a,b)P[a,\infty)(1 - P(b,\infty))$$

$$= P[a,b)P[a,\infty) = P[a,b) \ .$$

q.e.d.

x) $P[a,a) = 0$

Proof. $$P[a,a) = P(-\infty,a)(1 - P(-\infty,a))$$

$$= 0 \ .$$ q.e.d.

xi) If $a \leq b \leq c$, then

$$P[a,c] = P[a,b) \oplus P[b,c] \ .$$

Proof. As above we calculate

$$P[a,b)P[b,c] = 0$$

and

$$P[a,c] = P[a,b) + P[b,c] \ .$$

q.e.d.

Similarly,

$$P[a,c] = P[a,b] \oplus P(b,c] \ .$$

xii) If M is a finite factor and ${}^{tr}:M \to Z$ is the center valued trace, then

$$P[a,b)^{tr}(m) = P[a,b(m))^{tr}(m) \ .$$

<u>Proof</u>. Since K is Stonean, there exists an increasing sequence of simple functions $s_n \in \underline{R}^+(K)$ which converges uniformly to b. From v),

$$P[a,b) = \lim_{n\to\infty} P[a,s_n)$$

and by the continuity of ^{tr},

$$\lim_{n\to\infty} P[a,s_n)^{tr} = P[a,b)^{tr} .$$

On the other hand,

$$P[a,s_n)^{tr}(m) = P[a,s_n(m))^{tr}(m)$$

so that xii) follows.

xiii) If Q is a projection in M such that $Q \leq P[b,b]$, then Q commutes with $P[I]$ for all intervals I.

<u>Proof</u>. Without loss of generality we may assume that $I = [a,c]$, $a \leq c$. Let $z = \Omega(K)$ be a maximal central projection such that $zP[b,b] \leq zP(I)$. Then $(1-z)P[b,b]P(I) = 0$. But Q commutes with z since $Q \in M$. Hence Q commutes with $P(I)$.

Remark. $P(I)$ could be extended to a projection valued measure on the Borel subsets of $K \times \underline{R}$. This might unify the elementary properties i - xiii , but it would delay the presentation of our main result, Lemma 3.

Lemma 3. Let M be a W* algebra with a center valued trace

$$^{tr}:M \to Z .$$

Let P be the spectral family of $A = A^* \in M$ as usual. Let k be a positive integer and let a,b,c be three continuous real valued functions on K such that $a \leq b$ and

$$0 \leq c < \Phi(P[a,b)^{tr}) - \epsilon$$

for some $\epsilon > 0$. Suppose there exists a projection $S \in M$, $S \neq 0$, such that $S^{tr} = \frac{c}{k}$. Then there exists k equivalent orthogonal projections $P_j \in M$ and k continuous real valued functions a_j on K such that

1) the P_j commute with A and are equivalent to S,
2) $a \leq a_{j-1} \leq a_j \leq b$ where $a_0 = a$, and
3) $P(a_{j-1},a_j) \leq P_j \leq P[a_{j-1},a_j]$

for all $j \in \{1,\cdots,k\}$.

Proof. The proof is by induction on j. As our induction hypothesis we take the conclusions of the lemma and

$$4)\quad P[a,a_j) \leq \Sigma_{i=1}^{j} P_i \leq P[a,a_j] \ .$$

To get the induction started, we take $a_0 = a$ and $P_0 = 0$. (a_0,P_0) does not satisfy 1-4 for $j=0$ so that the induction hypothesis must be checked separately for the case $j=1$ in the following construction.

Fix j such that $1 \leq j \leq k$ and suppose that (a_i,P_i) have been constructed for $0 \leq i < j$.

Let $P' = PQ$ with $Q = 1 - \Sigma_{i=1}^{j-1} P_i$ as discussed in property iv). For $j=1$, $P' = P$ and $P'[a,a_{j-1}) = P[a,a) = 0$ by property x . For $j > 1$ and any interval I ,

$$P'(I) = P(I)(1 - \Sigma_{i=1}^{j-1} P_i) \ .$$

Hence $P'[a,a_{j-1}) = 0$ from hypothesis 4:

$$P[a,a_{j-1}) \leq \Sigma_{i=1}^{j-1} P_i \leq P[a,a_{j-1}] \ .$$

Furthermore, when $a_j \geq a_{j-1}$ is constructed, hypothesis 4) will imply

$$P'(a_{j-1},a_j) = P(a_{j-1},a_j)$$

and

$$P'[a_{j-1},a_j] \leq P[a_{j-1},a_j] \ .$$

We will construct (a_j,P_j) satisfying

1') $P_j \leq 1 - \Sigma_{i=1}^{j} P_i$ and P_j commutes with $P'(I)$ for all I and $P_j^{tr} = \frac{c}{k}$.

2') $a_{j-1} \leq a_j \leq b$,

3') P_j is orthogonal to P_i for $i < j$ and

4') $P'[a_{j-1}, a_j) \leq P_j \leq P'[a_{j-1}, a_j]$.

Hypothesis 1), 2), 3) and 4) follow easily from 1'), 2'), 3') , and 4') and the previous paragraph.

We define

$$a_j(m) = \sup \{d \in \underline{R} | p'[a,d)^{tr}(m) < \frac{c(m)}{k}\} .$$

Then $P'[a, a_{j-1}(m))^{tr}(m) = P'[a, a_{j-1})^{tr}(m)$ by property xii) so that $P'[a, a_{j-1}(m))^{tr}(m) = 0$ and $a_{j-1}(m)$ is an element of the set in the right hand side of the above equation. Hence $a_{j-1} \leq a_j$ which is the first part of hypothesis 2).

To show that a_j is continuous, we will first show that

$$a_j = \sup \{s | s \text{ is simple and } P'[a,s)^{tr} + \epsilon < \frac{c}{k} \text{ for some } \epsilon > 0\} .$$

This will establish that a_j is lower semicontinuous. Clearly the right hand side is less than or equal to a_j . To show that $a_j(m)$ is less than or equal to the right hand side (m fixed), we fix $d \in \underline{R}$ such that $P'[a,d)^{tr}(m) < \frac{c(m)}{k}$. $a_j(m)$ is the sup of such d .

There exists an $\epsilon > 0$ and a clopen set E containing m such that

$$P'[a,d)^{tr}(n) + \epsilon < \frac{c(n)}{k}$$

for all $n \in E$. Hence

$$P'[a, d\chi_E - \infty\chi_{E^c}) + \epsilon < \frac{c}{k}$$

where E^c is the complement of E (see property v). Hence $d\chi_E - \infty\chi_{E^c}$ is in the set in the right hand side of the equality in question and $d = (d\chi_E - \infty\chi_{E^c})(m)$ so the right hand side is greater than or equal to d. Hence the right hand side is greater than or equal to $a_j(m)$ for all $m \in K$ and our equality is established.

Similarly b_j is upper semi-continuous where

$$b_j(m) = \inf\ \{d \in \underline{R} | P'[a,d)^{tr}(m) > \frac{c(m)}{K}\ \}\ .$$

But $a_j(m) = b_j(m)$ since $P'[a,d)^{tr}(m)$ is a monotonic function of d. Hence a_j is continuous.

To establish the rest of 2), we will show that $b(m)$ is an element of the set in the above equality:

$$\begin{aligned} P'[a,b(m))^{tr}(m) &= (P[a,b) - \Sigma \oplus_{i=1}^{j-1} P_i)^{tr}(m) \\ &= P[a,b)^{tr}(m) - \frac{(j-1)\quad c(m)}{k} \\ &> \frac{c(m)}{k}\ . \end{aligned}$$

Hence $a_j(m) = b_j(m) \leq b(m)$.

We want to be able to choose

$$Q_j \leq P'[a_j, a_j]$$

such that $(P'[a_{j-1}, a_j) \oplus Q_j)^{tr} = \frac{c}{k}$. Then $P_j =$ $P'[a_{j-1}, a_j) \oplus Q_j$ will satisfy hypothesis 1′) (see property xiii), 3′) and 4′) .

We have that

$$P'[a_{j-1}, a_j)^{tr}(m) \leq \frac{c(m)}{k} \leq P'[a_{j-1}, d)^{tr}(m)$$

for all $d > a_j(m)$, $m \in K$, by the definition of $a_j(m)$ and property xii).

Hence by property viii),

$$P'[a_{j-1}, a_j]^{tr} - P'[a_{j-1}, a_j)^{tr}$$

$$\geq \frac{c}{k} - P'[a_{j-1}, a_j)^{tr} \geq 0 .$$

Since there exists a projection $S \in M$ such that $S^{tr} = \frac{c}{k}$, we may choose a projection $Q_j \leq P'[a_{j-1}, a_j]$ $- P'[a_{j-1}, a_j) = P'[a_j, a_j]$ such that

$$Q_j^{tr} = \frac{c}{k} - P'[a_{j-1}, a_j)^{tr} .$$

q.e.d.

CHAPTER IV

AN APPROXIMATION PROPERTY OF W* ALGEBRAS

In this chapter, we use the results of the previous chapter to derive an approximation property of W* algebras. Lemma 4.4 is the main result.

Let M be a W* algebra. Let A be a Hermitian element of M which is contained in no proper two sided ideal of M. Let $\{P(E)\}_{E\subset \underline{R}}$ be the spectral family of A.

Lemma 4.1. There exists a $\delta > 0$ such that $P(\{x \mid \delta \leq |x|\})$ is contained in no proper two sided ideal of M.

Proof. Suppose that for each positive integer n, the smallest two sided ideal J_n containing $P(\{x \mid \frac{1}{n} \leq |x|\})$ is a proper subset of M. Then the norm closure, J, of the union of the J_n is a proper subset of M since M has an identity. Furthermore, J is a two sided ideal since $J_n \subset J_{n+1}$, and $A \in J$ since

$AP(\{x|\frac{1}{n} \leq |x|\}) \to A$ in norm. In short

$$A \in \overline{\bigcup_{i=1}^{\infty} J_n} = J \subsetneqq M ,$$

which is a contradiction. q.e.d.

Lemma 4.2. Let P be a projection in a W* algebra M which is contained in no proper two sided ideal of M. Then there exist K partial isometries $W_i (i=1,\cdots,k)$ such that

$$1 = \Sigma_i \; W_i P \, W_i^* .$$

Proof. Since P is in no proper two sided ideal,

$$1 = \Sigma_{i=1}^{k} B_i P C_i$$

for some B_i, $C_i \in M$.

Let $B_i^* = U_i^* |B_i^*|$ be the polar decomposition of B_i^*. Then U_i is a partial isometry from the range projection $R(B_i^*)$ of B_i^* to the range projection of B_i, i.e.,

$$U_i : R(B_i^*) \sim R(B_i) .$$

Note that if D and E are positive then $R(DE) \sim R(ED)$ by polar decomposition. We have:

$$R(B_i P C_i) \leq R(|B_i^*| U_i P)$$
$$= R(|B_i^*| R(U_i P)) \sim R(R(U_i P) |B_i^*|)$$

$$\leq R(U_iP) = R(U_iR(B_i^*)P)$$

$$\sim R(R(B_i^*)P) \sim R(PR(B_i^*)) \leq P .$$

We will now construct W_i by induction on i. By the preceeding paragraph, there is a partial isometry $X_1 \in M$ from $R(B_1PC_1)$ into P. Let $W_1 = X_1^*$. Then

$$R(B_1PC_1) = W_1PW_1^* .$$

Suppose X_j $(j < i \leq k)$ have been constructed with $X_j^* X_j$ orthogonal to $X_h^* X_h$ for $i \neq h < i$ where X_j is a partial isometry from $R(B_jPC_j) - \Sigma_{h<j} X_h^* X_h$ into P. Let X_i be a partial isometry from $R(B_iPC_i) - \Sigma_{j<i} X_j^* X_j$ into P (end of induction). Set $W_i = X_i^*$. Then $W_jPW_j^* = W_jW_j^*$ is orthogonal to $W_iW_i^*$ for $j < i$ and

$$\bigvee_{i=1}^{k} R(B_iPC_i) \leq \Sigma_{i=1}^{k} W_iPW_i^* .$$

Finally,

$$1 = R(\Sigma_{i=1}^{k} B_iPC_i) \leq \bigvee_{i=1}^{k} R(B_iPC_i)$$

$$\leq \Sigma_{i=1}^{k} W_iPW_i^*$$

and $1 = \Sigma_{i=1}^{k} W_i PW_i^*$. q.e.d.

When M is a properly infinite W* algebra, Lemma 4.2

implies that a projection is contained in no proper two sided ideal of M if and only if it is equivalent to the identity.

When M is finite, Lemma 4.2 implies that a projection $P \in M$ is in no proper two sided ideal if and only if P^{tr} is invertible.

These facts imply Lemma 4.3:

<u>Lemma 4.3</u>. Take δ and $P(\{x|\delta < |x|\})$ from Lemma 4.1. Then there exists p equivalent orthogonal projections P_h $(h = 1,\cdots,p)$ and a projection $R \precsim P_h$ (R is for remainder) such that

i) $P(-\frac{\delta}{2}, \frac{\delta}{2}) = \Sigma_h P_h \oplus R$ and

ii) $P_h \precsim P(\{x|\delta < |x|\})$.

The proof of Lemma 4.4 requires the classification theory of W* algebras. We review the basic definitions here. M is <u>finite</u> if it has a center valued trace ${}^{tr}: M \to Z$ as defined in Chapter III. M is <u>type</u> II_1 if $\{P^{tr}|P$ is a projection$\} = \{z \in Z|0 \le z \le 1\}$. M is a <u>finite type</u> I algebra if M is finite but zM is not type II_1 for all projections $z \in Z$. M is <u>properly infinite</u> if zM is not finite for all non-zero projections $z \in Z$. Finally M is <u>type</u> I_n if $\{P^{tr}(m)|P$ a projection$\} = \{\frac{k}{n} \mid k = 0,\cdots,n\}$ $(m \in K)$.

We are now ready to prove the approximation property.

Lemma 4.4. For all Hermitian A in M contained in no proper two sided ideal of M, there exists a $\delta > 0$ and a positive integer m such that for all $\epsilon > 0$, $\epsilon < \frac{\delta}{2}$, there exists a subalgebra N of M and a Hermitian element $B \in N$ such that

i) $\|(A-B)1_N\| \leq \epsilon$ where 1_N is the projection in M which is the identity in N,

ii) 1_N commutes with A,

iii) $A(1 - 1_N)$ is invertible in $(1-1_N)M(1-1_N)$,

iv) $\delta \leq |a|$ for all non-zero eigenvalues a of B,

v) N is a direct sum of type $I_n W^*$ algebras (where $n < \infty$ vaires over a finite set) with trace

$$^{tr}:N \to Z ,$$

and

vi) $(\text{kernel } B)^{tr} \leq m\ (\text{range } B)^{tr}$.

Proof. M is the direct sum of a type II_1, a finite type I, and a properly infinite W* algebra. Hence Lemma 4.4 follows from these special cases by taking direct sums. We will treat the case when M is type I_n $(n < \infty)$ first as it is the easiest and will be useful in the finite type I case.

Let $\delta > 0$ and $m = 4p$ be taken from Lemmas 4.1

and 4.3. m may be chosen smaller than this in some cases and we will do so for the sake of exposition.

Recall that $\{P(E)\}_{E\subset \underline{R}}$ is the spectral family of A.

<u>Case I</u>: M <u>is type</u> I_n $(n < \infty)$.

This implies $M = Z \otimes \underline{B}(\underline{C}^n)$ (see Sakai [6]). Let $m = p+1$. Let $Q = P(\{x | \epsilon > |x|$ or $\delta < |x|\})$. Then $N = QMQ$ is a direct sum of type I_K algebras $(k \leq n)$ of M (condition v) and

$$B = AP(\{x | \delta < |x|\}) \in N .$$

i) $\|(A-B)1_N\| < \epsilon$, ii) $1_N = Q$ commutes with A , iii) $A(1-1_N) = AP(\{x | \epsilon \leq |x| \leq \delta\})$ is invertible in $(1-1_N)M(1-1_N)$, and iv) $\delta \leq |a|$ for all non-zero eigenvalues a of B .

The conclusions of Lemma 4.3,

$$P(\{x | \tfrac{\delta}{2} > |x|\}) = \Sigma \oplus P_i \oplus R \quad (i=1,\cdots,m-1) \quad \text{and}$$
$$R \precsim P_i \sim P_j \precsim P(\{x | \delta < |x|\})$$

as $i,j = 1,\cdots,p$, imply that

$$\frac{P(\{x | \epsilon > |x|\})^{tr}}{m} \leq P(\{x | \delta < |x|\})^{tr}$$

which is condition vi)

<u>Case II</u>: M <u>is properly infinite</u>.

In this case, we may take $m = p = 1$ and $R = 0$.

Let $\{[a_{i-1}, a_i]\}_{i=1}^{k}$ be a partition of $[-\|A\|, \delta] \cup [\delta, \|A\|]$ with mesh less than ϵ. By repeated application of the comparability theorem, we can find ℓ orthogonal central projections z ; $(j=1,\cdots,\ell)$ such that $\Sigma \oplus_j z_j = 1$ and $\{z_j P[a_{i-1}, a_i]\}_{i=1}^{k}$ is totally ordered by $\lesssim$ for each j. Let $z_j P[a_{j-1}, a_j]$ be the maximal element in $z_j P[a_{i-1}, a_i] (i=1,\cdots,k)$. Then $z_j P[a_{j-1}, a_j]$ is contained in no proper two sided ideal of $z_j M$.

$z_j P[a_{j-1}, a_j]$ is equivalent to z_j (Lemma 4.3) and z_j is infinite. Hence there exists a partial isometry U_j from $z_j P[-\epsilon, \epsilon]$ into (and not onto) $z_j P[a_{j-1}, a_j]$.

We let N_j be the subalgebra of M generated by U_j and $z_j P[a_{j-1}, a_j] - U_j U_j^*$.

$$N_j \cong \underline{B}(\underline{C}^2) \oplus \underline{C}$$

so that N_j is the direct sum of two types I_n W* algebras. We let

$$N = \Sigma_j \ N_j$$

and $B = \Sigma_j \ a_j \ z_j P[a_{j-1}, a_j]$. Then B and N satisfy i) - v) and $B \in N$. The matrix of $z_j B$ in N_j is

$$\begin{matrix} 0 & 0 & 0 \\ 0 & a_j & 0 \\ 0 & 0 & a_j \end{matrix} .$$

Case III: M is type II_1 .

Let $m = 2p+1$. Divide $P[-\epsilon,\epsilon]$ into m equivalent orthogonal projections P_h $(h=1,\cdots,m)$ with $P[-\epsilon,\epsilon] = \Sigma_h P_h$ and

$$P_h^{tr} = \frac{P[-\epsilon,\epsilon]^{tr}}{m} .$$

Let k be a positive integer which is large enough so that

$$k\ \epsilon > 6\|A\| .$$

Each P_h may be divided into k equivalent orthogonal projections, $P_{h,i}$ $(i=1,\cdots,k)$ with $P[-\epsilon,\epsilon] = \Sigma_{h,i} P_{h,i}$ and

$$P_{h,i}{}^{tr} = \frac{P[-\epsilon,\epsilon]^{tr}}{mk} .$$

To apply Lemma 3, we first note that

$$\frac{2P[-\epsilon,\epsilon]^{tr}}{m} \leq P(\{x|\delta \leq |x|\})^{tr} - \epsilon'$$

for some $\epsilon' > 0$. Next we note that there exists a projection $S \in M$ with $S^{tr} = \frac{P[-\epsilon,\epsilon]^{tr}}{km}$. In fact, $S = P_{1,1}$ will suffice.

Hence we may apply Lemma 3 to $P' = PP(\{x|\delta < |x|\})$ (see property iv) with $c = \Phi(\frac{2P[-\epsilon,\epsilon]^{tr}}{m})$, $b = \|A\| + 1$, $a = -\|A\|$, and k replaced by $2k$ to obtain $2k$ continuous functions $a_j \in \underline{R}(K)$ and $2k$ equivalent orthogonal projections $P_i' \in M$ $(i=1,\cdots,2k)$ such that

1) the P_i' commute with $P'(I)$ for all intervals I and $P_i'^{\,tr} = \frac{P[-\epsilon,\epsilon]^{tr}}{mk}$

2) $a \leq a_{i-1} \leq a_i \leq b$ where $a_0 = a$ and

3) $P'(a_{i-1},a_i) \leq P_i' \leq P'[a_{i-1},a_i]$.

Since $P'(\{x|\delta > |x|\}) = P'(\|A\|,\infty) = 0$, we may assume $\delta \leq |a_i| \leq \|A\|$ $(i=1,\cdots,2k)$ without loss of generality. Hence the P_i' commute with A (see property iv).

Approximate the a_i uniformly by simple functions in $\underline{R}(K)$,

$$\sup_K |a_i - \Sigma_j a_i{}^j \chi_{E_j}| < \frac{\epsilon}{3} \quad (j=1,\cdots,\ell)$$

where the $a_i{}^j$ are chosen so that

$$\delta \leq |a_i{}^j| \leq \|A\| \quad \text{and} \quad a_{i-1}^j \leq a_i{}^j .$$

Set $z_j = \Omega(\chi_{E_j})$ $(j=1,\cdots,\ell)$. Then

$$z_j P_i' \leq P[a_{i-1}^j - \frac{\epsilon}{3} , a_i^j + \frac{\epsilon}{3}] \quad \text{and}$$

$$1 = \Sigma_j z_j .$$

Fix j . At least half of the intervals $[a_{i-1}^j, a_i^j]$ $(i=1,\cdots,2k)$ have length less than $\frac{\epsilon}{3}$ for otherwise, at least half of them have length $\geq \frac{\epsilon}{3}$ and

$$\Sigma_{j=1}^{2k} a_i^j - a_{i-1}^j \geq \frac{k\epsilon}{3} > 2\|A\|$$

which contradicts our choice of the a_i^j. Take k of the $z_j P_i'$ with $a_i^j - a_{i-1}^j < \frac{\epsilon}{3}$ and call them $z_j P_{n,i}$ $(n = m+1,\ i=1,\cdots,k)$. Then $\|Az_j P_{h,i}\| < \epsilon$ and $z_j P_{h,i}$ are all equivalent. $(i=1,\cdots,k,\ h=1,\cdots,n)$. Hence for each j they generate a factor N_j of type I_{kn}. Let

$$N = \Sigma_j N_j$$

and

$$B = \Sigma_{i,j}\, a_i^j\, z_j\, P_{n,i}\ .$$

Then $B \in N$ and i) - v) are clearly satisfied. For vi), we use that for each j, $z_j P_{h,i}$ $(h=1,\cdots,n\ ,\ i=1,\cdots,k)$ are equivalent. Furthermore,

$$\text{range } B = \Sigma_{i,j}\, z_j P_{n,i}$$

and kernel $B = \Sigma_{i,j,h}\, z_j P_{h,i}$ where $h = 1,\cdots,m$. Hence

$$(\text{kernel } B)^{tr} = m(\text{range } B)^{tr}$$

and vi) follows.

Case IV: M is finite type I.

This case is essentially the same as Case III. We will outline the differences.

Case I may be applied to the first q terms of the decomposition

$$M = \Sigma \oplus_{n=1}^{\infty} z_n Z \otimes \underline{B}(\underline{C}^n)$$

to insure that $P(\{x|\delta < |x|\})$ may be divided into k parts. Also dividing $P[-\epsilon,\epsilon]$ into mk (or fewer) parts may leave a remainder. This may be taken care of by dividing $P(\{x|\delta < |x|\})$ into 2 parts, putting the remainder in the second part, and the rest in the first part. This requires taking $m = 4p$ to apply Lemma 3. To divide $P(\{x|\delta < |x|\})$ into $2k$ parts, we need $q > \frac{2k}{f}$ where f is the minimum value of $\Phi(P(\{x|\delta < |x|\})^{tr})$. Since we require $k\,\epsilon > 6\|A\|$ as in Case III, it suffices to choose

$$q > \frac{12\|A\|}{\epsilon f}$$

to insure that $P(\{x|\delta < |x|\})$ may be divided as desired. It is not possible to choose q to insure that $P[-\epsilon,\epsilon]$ may be divided into mk parts, but nothing is lost by dividing it into fewer parts. q.e.d.

CHAPTER V

THE SOLUTION FOR A W* ALGEBRA

We first prove a refined version of Theorem B for type I_n algebras.

<u>Lemma 5.1</u>. Let $N = Z \otimes \underline{B}(\underline{C}^n)$ be a type I_n algebra $(n < \infty)$ with center Z. Let B be a Hermitian element of N which is contained in no proper two sided ideal of N. Then there exists a Hermitian T in N such that $\sigma(T+iB) \cap \underline{R} = \emptyset$.

Furthermore, suppose that for some positive integer m and some $\delta > 0$,

$$(\text{kernel } B)^{tr} \leq m(\text{range } B)^{tr}$$

and

$$\delta < \inf \{|a| \mid a \neq 0 \text{ and } a \in \sigma(B)\} .$$

Then T may be chosen so that

$$\sup_t \|(T+iB-t)^{-1}\| < d(m,\delta)$$

where $d(m,\delta)$ is a constant depending on m and δ but not on iB or N .

<u>Proof</u>. Let $k = \max\{d_h | h=1,\cdots,m+1\}$ where

$$d_h = \sup_t \|(T_h+iP_h-t)^{-1}\|$$

as defined in Chapter II. Let

$$d(m,\delta) = \frac{k}{\delta} .$$

Let K be the maximal ideal space of Z . Then Z is isomorphic to the continuous complex valued functions, $\underline{C}(K)$ on K and N is isomorphic to the algebra of $n \times n$ matrices with entries from $\underline{C}(K)$ (Sakai [6] 1.22.10, 1.22.12, and 2.3.2). Using the spectral decomposition of B , we may choose a basis for N so that B is represented by a diagonal matrix.

With these identifications, the center valued trace is given by

$$n(C)^{tr}(t) = \text{trace } (C(t))1_N$$

where $C \in N$ and $C(t)$ is the matrix obtained by evaluating the entries of C at $t \in K$. Hence $(\text{range } B)^{tr}(t)$ is the number of non-zero entries in $B(t)$ and $(\text{kernel } B)^{tr}(t)$ is the number of zeros along the diagonal of $B(t)$.

For each $t \in K$, $B(t)$ has at least one non-zero entry since B is contained in no proper two sided ideal of N. Furthermore for every m zeros along the diagonal of $B(t)$, there is a non-zero entry. Hence we may choose a basis for N so that along the diagonal $B(t)$ has at most m zeros followed by a non-zero element followed by at most m zeros followed by a non-zero element and so on.

More precisely, we first partition K into n sets $\{K_i\}_{i=0}^{n}$ with

$$n(\ker B)^{tr}(t) = i \quad \text{for} \quad t \in K_i \ .$$

We then divide the matrix $\chi_{K_i} B$ into ℓ $(m+1)$ by $(m+1)$ matrices, one k_i by k_i matrix, and $n - (k+\ell(m+1))$ one by one matrices where χ_{K_i} is the characteristic function of K_i, ℓ is the greatest integer less than $\frac{i}{m}$ and $k_i - 1$ is the remainder $i - \ell m$. Hence $\chi_{K_i} B$ is partitioned into the direct sum of $k_{i,j}$ by $k_{i,j}$ matrices $B_{i,j}$ where $k_{i,j} \leq m+1$:

$$B = \sum_{i,j} B_{i,j} \ .$$

We choose the basis for $\chi_{K_i} N$ so that each matrix $B_{i,j}$ has an entry $a_{i,j}$ in the lower right hand corner which vanishes nowhere on K_i and zeros elsewhere. In other words if $k = k_{i,j}$,

$$B_{i,j} = a_{i,j} P_k$$

where P_k is the matrix of Lemma 2.1. Let $T_{i,j} = a_{i,j}T_k$ in the basis for $B_{i,j}$ and let

$$T = \Sigma_{i,j} T_{i,j} .$$

Finally $\sigma(B) \cup \{0\}$ is the union of $\{0\}$ with the ranges of the $a_{i,j}$. Hence $|a_{i,j}(s)| > \delta$ for all $s \in K_i$ implies that

$$\begin{aligned}
&\sup_t \|(T+iB-t)^{-1}\| \\
&= \sup_{i,t,j} \sup_{s \in K_i} \|(a_{i,j}(s)(T_k+iP_k-t))^{-1}\| \\
&\leq \frac{1}{\delta} \sup_h d_h = d(m,\delta) .
\end{aligned}$$

where $h = 1,\cdots,m+1$ and $k = k_{i,j} \leq m+1$. q.e.d.

<u>Theorem B</u>. Let M be a W* algebra. Let $A = A^* \in M$ be contained in no proper two sided ideal of M. Then there exists $T = T^* \in M$ such that

$$\sigma(T+iA) \cap \underline{R} = \emptyset .$$

<u>Proof</u>. Take $\delta > 0$ and $m = 4p$ from the approximation property, Chapter IV. Let $\epsilon > 0$, and $\epsilon < \frac{1}{d(m,\delta)}$. Then the approximation property supplies us with a $B \in N$ fulfilling the hypothesis of Lemma 5.1 as well as

satisfying

i) $\|(A-B)1_N\| < \epsilon$,

ii) 1_N commutes with A , and

iii) $A(1-1_N)$ is invertible in $(1-1_N)M(1-1_N)$.

Applying Lemma 5.1 to $B \in N$, we obtain a $T \in N$ such that

$$\sup_{t\in\underline{R}} \|(T+iB-t)^{-1}\| < \epsilon .$$

Since $1_N T = T$ and 1_N commutes with A ,

$$T+iA = iA(1-1_N) \oplus T+iA1_N .$$

Hence the spectrum of $T+iA$ is the union of the spectrum of $iA(1-1_N)$ in $(1-1_N)M(1-1_N)$ and the spectrum of $T+iA1_N$ in N .

Now $A(1-1_N)$ is an invertible Hermitian element of $(1-1_N)M(1-1_N)$ so that

$$\sigma(iA(1-1_N)) \cap \underline{R} = \emptyset$$

in $(1-1_N)M(1-1_N)$.

On the other hand,

$$\|(T+iA1_N) - (T+iB)\|$$
$$= \|(A-B)1_N\| \leq \epsilon < \frac{1}{d(m,\delta)}$$

and

$$\sup \|(T+iB-t)^{-1}\| < d(m,\delta)$$

so that

$$\sigma(T+iA1_N) \cap \underline{R} = \emptyset$$

in N by Lemma 2.2. Hence

$$\sigma(T+iA) \cap \underline{R} = \emptyset .$$

q.e.d.

<u>Remark</u>. The above proof shows that Theorem B is true for any C* algebra M where an approximation property holds. This does not require the spectral family of A to actually be in M. We will prove Theorem 2 in the next chapter by showing that the shift algebra satisfies an approximation property.

<u>Theorem C</u>. Let $A \in M$ be an element of a W*algebra M. Suppose that A is contained in no proper two sided ideal of M. Then there exists a $T \in M$ such that

$$\sigma(T+A) \cap \sigma(T) = \emptyset .$$

<u>Proof</u>. Let $A = A_1 + iA_2$ be the Hermitian decomposition of A. Let $\{P_1(E)\}_{E\subset\underline{R}}$ and $\{P_2(E)\}_{E\subset\underline{R}}$ be the spectral families of A_1 and A_2 respectively. Since

$$A = \lim_{\epsilon\to 0^+} [A_1P_1(\{x|\ \epsilon < |x|\}) + iA_2P_2(\{x|\epsilon < |x|\})]$$

in the uniform topology and A is contained in no proper two sided ideal, there exists an $\epsilon > 0$ such that the ideal generated by $P_1(\{x|\epsilon < |x|\}) = Q_1$ and $P_2(\{x|\ \epsilon < |x|\}) = Q_2$

is all of M . By the comparability theorem, there exists a central projection z such that

$$zQ_1 \lesssim zQ_2$$

and

$$(1-z)Q_1 \gtrsim (1-z)Q_2 \ .$$

Hence zA_2 and $(1-z)A_1$ are contained in no proper two sided ideal of zM and $(1-z)M$ respectively. By Theorem B, there exists a $T_2 = T_2^* \in zM$ and a $T_1 = T_1^* \in (1-z)M$ such that

$$\sigma(z(T_2+iA_2)) \cap \underline{R} = \emptyset$$

and

$$\sigma((1-z)(-iT_1+A_1)) \cap i\underline{R} = \emptyset$$

in zM and $(1-z)M$ respectively. Let

$$T = z(k_2+T_2-A_1) \oplus i(1-z)(k_1-T_1-A_2)$$

where k_1 , $k_2 \in R$ are large enough so that

$$\sigma(z(k_2+T_2+iA_2)) \cap i\underline{R} = \emptyset$$

and

$$\sigma((1-z)(i(k_1-T_1) + A_1)) \cap \underline{R} = \emptyset$$

in zM and $(1-z)M$ respectively. Then

$$\sigma(T) \subset \underline{R} \cup i\underline{R}$$

and

$$T+A = z(k_2+T_2+iA_2)$$

$$\oplus(1-z)(i(k_1-T_1) + A_1)$$

so that

$$\sigma(T+A) \cap (\underline{R} \cup i\underline{R}) = \emptyset$$

and

$$\sigma(T) \cap \sigma(T+A) = \emptyset .$$

q.e.d.

CHAPTER VI

THE SOLUTION FOR THE SHIFT ALGEBRA

The converse of Theorem A fails in the shift algebra for the unilateral shift (Theorem 3). The unilateral shift is not contained in any proper two sided ideal since it is Fredholm, i.e., the range is closed, the kernel has finite dimension, and the codimension of the range is finite (Theorem 1). Furthermore, the analogue of Theorem B is true in the shift algebra (Theorem 2). The proofs of these results use the representation of the unilateral shift as a multiplication operator on the Hardy space $H^2(S^1)$.

Let $L^2(S^1)$ be the Hilbert space of L^2 functions on the unit circle with respect to Lebesgue measure. Let $H^2(S^1)$ be the L^2 functions whose negative Fourier coefficients are zero. Let P be the orthogonal projection of $L^2(S^1)$ onto $H^2(S^1)$. For a continuous complex valued function $f(f \in \underline{C}(K))$, we define $T_f \in \underline{B}(H^2(S^1))$ by

$$T_f(q) = P(f \cdot q)$$

where $\cdot$ is pointwise multiplication and $q \in H^2(S^1)$. T_g is the unilateral shift if we define $g \in \underline{C}(S^1)$ by $g(\theta) = e^{i\theta}$ (we view S^1 as the reals mod 2π). The shift algebra M is the smallest C^*subalgebra of $B(H^2(S^1))$ containing T_g. There is an exact sequence of C^* algebra homomorphisms

$$0 \to K \xrightarrow{i} M \xrightarrow{\pi} \underline{C}(S^1) \to 0$$

where K is the compact operators in $\underline{B}(H^2(S^1))$, i is inclusion, and $\pi(T_f) = f$ for all $f \in \underline{C}(S^1)$ (Douglas [1], Theorem 1, p. 7).

Theorem 1. An element A in M is contained in no proper two sided ideal of M if and only if A is Fredholm.

Proof. Every nonzero closed two sided ideal of M contains the compacts. Also the closure of a proper two sided ideal of M is proper since M has an identity. Hence A is contained in no proper two sided ideal of M if and only if $\pi(A)$ is contained in no proper two sided ideal of $\underline{C}(S^1)$ and this is true if and only if $\pi(A)$ is invertible. But π induces $\underline{C}(S^1) \cong M/K$ so that $\pi(A)$ is invertible if and only if A is Fredholm by Atkinson's theorem [4]. q.e.d.

Theorem 2. Given $A = A^* \in M$, A contained in no proper two sided ideal of M, there exists a $T = T^* \in M$ such that

$$\sigma(T+iA) \cap \underline{R} = \emptyset .$$

Proof. We will show M satisfies Lemma 4.4. The proof of Theorem B carries over to obtain T with $\sigma(T+iA) \cap \underline{R} = \emptyset$.

So fix $A = A^* \in M$ contained in no proper two sided ideal of M . Let $\{P(E)\}_{E\subset\underline{R}}$ be the spectral family of A , i.e.,

$$A = \int_{-\infty}^{\infty} x dP(x) .$$

A is Fredholm by Theorem 1. Hence $A/_{(\ker A)^\perp}$ is invertible and

$$A = \int_{-\infty}^{-\delta} + \int_{\delta}^{\infty} x dP(x)$$

for some $\delta > 0$ since $P(\{0\})$ is the projection of $H^2(S^1)$ onto the kernel of A . ($P(\{0\})$ is finite dimensional.)

Fix $\epsilon > 0$, $\epsilon < \frac{\delta}{2}$. We will construct a subalgebra N of M and a $B = B^* \in N$ which satisfy the conclusions i) - vi) of Lemma 4.4.

Let $a \in \underline{R}$ be such that $[a,a+\epsilon] \subset (-\infty,-\delta] \cup [\delta,\infty)$ and $\dim P[a,a+\epsilon] = \infty$. Such an a exists since $P((-\infty,-\delta] \cup [\delta,\infty))$ is infinite dimensional.

Let U be a partial isometry from $P(\{0\})$ into $P[a,a+\epsilon]$. U is compact so that $U \in M$. Let N_1 be the C* algebra generated by U . Then N_1 is isomorphic to the two by two matrices. Let N_2 be the C*algebra generated by $P[a,a+\epsilon] - UU^*$. Then

$$N = N_1 \oplus N_2 \cong M_2 \oplus \underline{C}$$

and $P[a,a+\epsilon] \in N \subset M$. Let

$$B = aP[a,a+\epsilon] .$$

The matrix of B is

$$\left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & a & 0 \\ \hline 0 & 0 & a \end{array}\right) \in M_2 \oplus \underline{C} .$$

i) $1_N = P(\{0\}) \oplus P[a,a+\epsilon]$ commutes with A ,

ii) $\|(A-b)1_N\| = \|(A-aP[a,a+\epsilon])1_N\| \leq \epsilon$,

iii) $A(1-1_N)$ is invertible in $(1-1_N)M(1-1_N)$,

iv) $\delta \leq |a|$ for the only non-zero eigenvalue a of B ,

v) N is a direct sum of M_2 and $\underline{C}$ with center $\underline{C} \oplus \underline{C}$. For $a \oplus b \in N$,

$$(a \oplus b)^{tr} = (\text{trace } a) \oplus b ,$$

and

vi) $(\ker B)^{tr} = \frac{1}{2} \oplus 0$ and $(\text{range } B)^{tr} = \frac{1}{2} \oplus 1$.

Hence

$$(\text{kernel } B)^{tr} < (\text{range } B)^{tr} .$$

We need some more structure for the proof of Theorem 3. First of all there is a continuous linear lifting, s, $\pi \circ s = id$, for the exact sequence

$$0 \to K \xrightarrow{i} M \xrightarrow{\pi} \underline{C}(K) \to 0$$

defined by $s(f) = T_f$. Secondly we have the following relation between the index of the Fredholm operator $T_f - x$ $(f \in \underline{C}(S^1)$, $x \in \underline{C} \setminus f(S^1))$ and the winding number $W_x(f)$ of f around x:

$$\text{index } (T_f - x) = -W_x(f)$$

where

$$\text{index } A = \dim \ker A - \text{codim range } A$$

for any Fredholm operator A, and

$$W_x(f) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f'(\theta)}{f(\theta) - x} \, d\theta$$

for any differentiable function f. This implies that $\sigma(T_f) \subset \sigma(T_f + B)$ for any compact operator $B \in K$, and

$$\sigma(T_f) = f(S^1) \cup \{x \mid W_x(f) \neq 0\} .$$

(See Douglas [1], Theorem 2, p. 9.)

<u>Theorem 3</u>. For all $X \in M$,

$$\sigma(X + T_g) \cap \sigma(X) \neq \emptyset$$

where T_g is the unilateral shift.

Proof. Fix $X \in M$. From the exact sequence introduced earlier,

$$X = T_f + B$$

where $f = \pi(X)$ and $B = X - T_f \in K$.

Let $h = f+g$. We wish to show that

$$\sigma(T_h+B) \cap \sigma(T_f+B) \neq \emptyset .$$

By the remarks preceeding the theorem, it suffices to show that

$$[h(S^1) \cup \{x|W_x(h) \neq 0\}] \cap [f(S^1) \cup \{x|W_x(f) \neq 0\}]$$

is non-null. If the above intersection is null, then

1) $f(S^1) \cap h(S^1) = \emptyset$
2) $W_{f(y)}(h) = 0$ for all $y \in S^1$ and
3) $W_{h(x)}(f) = 0$ for all $x \in S^1$.

We will show that conditions 1), 2) and 3) are inconsistent by calculating the integral of $\frac{h'(x) - f'(y)}{h(x) - f(y)}$ over the torus $T^2 = S^1 \times S^1$ in two different ways. First we separate $h(S^1)$ and $f(S^1)$ by open sets and homotop h and f to differentiable functions within these open sets so that conditions 1), 2) and 3) are still satisfied. In other words, we may assume that h and f are differentiable without loss of generality. Then

$$\int_{T^2} \frac{h'(x) - f'(y)}{h(x) - f(y)} \, dm(x,y)$$

$$= \int_0^{2\pi}\int_0^{2\pi} \frac{h'(x) - f'(y)}{h(x) - f(y)} \, dxdy$$

$$= 2\pi i \int_0^{2\pi} W_{f(y)}(h)\,dy + 2\pi i \int_0^{2\pi} W_{h(x)}(f)\,dx$$

$$= 0$$

where the Fubini theorem was used ([5], p. 140). For the other calculation, we introduce the differentiable transformation $U:C \to T^2$ defined by $(x,y) = U(t,s)$ where

$$x = t-s$$

$$y = t+s$$

and $C = \{(t,s) \in T^2 | 0 < s < \pi \,,\ 0 < t < 2\pi\}$ and T^2 is the Cartesian product of the reals mod 2π with themselves. The Jacobian of U is 2 so that by the change of variables formula ([5], p. 174)

$$\int_{T^2} \frac{h'(x) - f'(y)}{h(x) - f(y)} \, dm(x,y)$$

$$= \int_0^{\pi}\int_0^{2\pi} \frac{h'(t-s) - f'(t+s)}{h(t-s) - f(t+s)} \, 2 \, dt \, ds$$

$$= \int_0^{\pi} 4\pi i \, ds = 4\pi^2 i \, .$$

where $W_0(h(\cdot - s) - f(\cdot + s)) = 1$ since $t \to h(t-s) - f(t+s)$ is homotopic to $h-f = g$ in $\underline{C}\backslash\{0\}$, and

$W_0(g) = 1$. q.e.d.

Remarks. The integral in the above proof (due to Dr. L. Eifler) is a cohomology version of the original proof which applied the first homology functor H_1 to the diagram

$$\begin{tikzcd} S^1 \arrow[r, "\Delta"] \arrow[rrr, bend right, "g"'] & S^1 \times S^1 \arrow[r, "h\times f"] & \underline{C}\backslash C_{h,f} \times \underline{C}\backslash C_{f,h} \arrow[r, "-"] & \underline{C}\backslash\{0\} \end{tikzcd}$$

where $C_{h,f}$ is the component of $\underline{C}\backslash h(S^1)$ containing $f(S^1)$, Δ is the diagonal map $\Delta(x) = (x,x)$, $h\times f(x,y) = (h(x),f(y))$, and $-\ (x,y) = x-y$. The point is that $H_1(g) \neq 0$ but

$$\begin{aligned} H_1(h\times f) &= H_1(h) \otimes H_0(f) + H_0(h) \otimes H_1(f) \\ &= 0 \end{aligned}$$

since $H_1(h) = H_1(f) = 0$ ([3], p. 193-198).

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